TASI Lectures on Solitons

Lecture 3: Vortices

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Abstract: In this third lecture we turn our attention to vortex strings in non-abelian gauge theories. Once again we describe the various properties of the solutions and a D-brane construction of the vortex moduli space is reviewed. In this lecture we will see how instantons can be viewed from the worldsheet of a vortex string, an important result that underlies the relationship between 2d sigma-models and 4d gauge theories. We end with the usual vignettes, describing applications of vortices to cosmic strings and mirror symmetry.

Contents

3. Vortices

In this lecture, we're going to discuss vortices. The motivation for studying vortices should be obvious: they are one of the most ubiquitous objects in physics. On table-tops, vortices appear as magnetic flux tubes in superconductors and fractionally charged quasi-excitations in quantum Hall fluids. In the sky, vortices in the guise of cosmic strings have been one one of the most enduring themes in cosmology research. With new gravitational wave detectors coming on line, there is hope that we may be able to see the distinctive signatures of these strings as the twist and whip. Finally, and more formally, vortices play a crucial role in determining the phases of low-dimensional quantum systems: from the phase-slip of superconducting wires, to the physics of strings propagating on Calabi-Yau manifolds, the vortex is key.

As we shall see in detail below, in four dimensional theories vortices are string like objects, carrying magnetic flux threaded through their core. They are the semi-classical cousins of the more elusive QCD flux tubes. In what follows we will primarily be interested in the dynamics of infinitely long, parallel vortex strings and the long-wavelength modes they support. There are a number of reviews on the dynamics of vortices in four dimensions, mostly in the context of cosmic strings [1, 2, 3].

3.1 The Basics

In order for our theory to support vortices, we must add a further field to our Lagrangian. In fact we must make two deformations

- We increase the gauge group from $SU(N)$ to $U(N)$. We could have done this before now, but as we have considered only fields in the adjoint representation the central $U(1)$ would have simply decoupled.

- We add matter in the fundamental representation of $U(N)$. We'll add N_f scalar fields q_i, $i = 1 \ldots, N_f$.

The action that we'll work with throughout this lecture is

$$S = \int d^4x \, \mathrm{Tr} \left(\frac{1}{2e^2} F^{\mu\nu} F_{\mu\nu} + \frac{1}{e^2} (\mathcal{D}_\mu \phi)^2 \right) + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2$$

$$- \sum_{i=1}^{N_f} q_i^\dagger \phi^2 q_i - \frac{e^2}{4} \, \mathrm{Tr} \, (\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \, 1_N)^2 \tag{3.1}$$

The potential is of the type admitting a completion to $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetry. In this context, the final term is called the D-term. Note that everything in the bracket

of the D-term is an $N \times N$ matrix. Note also that the couplings in front of the potential are not arbitrary: they have been tuned to critical values.

We've included a new parameter, v^2, in the potential. Obviously this will induce a vev for q. In the context of supersymmetric gauge theories, this parameter is known as a Fayet-Iliopoulos term.

We are interested in ground states of the theory with vanishing potential. For $N_f < N$, one can't set the D-term to zero since the first term is, at most, rank N_f, while the v^2 term is rank N. In the context of supersymmetric theories, this leads to spontaneous supersymmetry breaking. In what follows we'll only consider $N_f \geq N$. In fact, for the first half of this section we'll restrict ourselves to the simplest case:

$$N_f = N \tag{3.2}$$

With this choice, we can view q as an $N \times N$ matrix $q^a{}_i$, where a is the color index and i the flavor index. Up to gauge transformations, there is a unique ground state of the theory,

$$\phi = 0 \quad , \quad q^a{}_i = v\delta^a{}_i \tag{3.3}$$

Studying small fluctuations around this vacuum, we find that all gauge fields and scalars are massive, and all have the same mass $M^2 = e^2 v^2$. The fact that all masses are equal is a consequence of tuning the coefficients of the potential.

The theory has a $U(N)_G \times SU(N)_F$ gauge and flavor symmetry. On the quark fields q this acts as

$$q \to UqV^\dagger \qquad U \in U(N)_G, \quad V \in SU(N)_F \tag{3.4}$$

The vacuum expectation value (3.3) is preserved only for transformations of the form $U = V$, meaning that we have the pattern of spontaneous symmetry breaking

$$U(N)_G \times SU(N)_F \to SU(N)_{\text{diag}} \tag{3.5}$$

This is known as the color-flavor-locked phase in the high-density QCD literature [4].

When $N = 1$, our theory is the well-studied abelian Higgs model, which has been known for many years to support vortex strings [5, 6]. These vortex strings also exist in the non-abelian theory and enjoy rather rich properties, as we shall now see. Let's choose the strings to lie in the x^3 direction. To support such objects, the scalar fields q must wind around $\mathbf{S}^1_\infty$ at spatial infinity in the (x^1, x^2) plane, transverse to the string.

As we're used to by now, such winding is characterized by the homotopy group, this time

$$\Pi_1\left(U(N) \times SU(N)/SU(N)_{\text{diag}}\right) \cong \mathbf{Z} \qquad (3.6)$$

which means that we can expect vortex strings supported by a single winding number $k \in \mathbf{Z}$. To see that this winding of the scalar is associated with magnetic flux, we use the same trick as for monopoles. Finiteness of the quark kinetic term requires that $\mathcal{D}q \sim 1/r^2$ as $r \to \infty$. But a winding around $\mathbf{S}^1_\infty$ necessarily means that $\partial q \sim 1/r$. To cancel this, we must turn on $A \to i\partial q\, q^{-1}$ asymptotically. The winding of the scalar at infinity is determined by an integer k, defined by

$$2\pi k = \text{Tr} \oint_{\mathbf{S}^1_\infty} i\partial_\theta q\; q^{-1} = \text{Tr} \oint_{\mathbf{S}^1_\infty} A_\theta = \text{Tr} \int dx^1 dx^2\; B_3 \qquad (3.7)$$

This time however, in contrast to the case of magnetic monopoles, there is no long range magnetic flux. Physically this is because the theory has a mass gap, ensuring any excitations die exponentially. The result, as we shall see, is that the magnetic flux is confined in the center of the vortex string.

The Lagrangian of equation (3.1) is very special, and far from the only theory admitting vortex solutions. Indeed, the vortex zoo is well populated with different objects, many exhibiting curious properties. Particularly interesting examples include Alice strings [8, 9], and vortices in Chern-Simons theories [10]. In this lecture we shall stick with the vortices arising from (3.1) since, as we shall see, they are closely related to the instantons and monopoles described in the previous lectures.

Figure 1:

To my knowledge, the properties of non-abelian vortices in this model were studied only quite recently in [11] (a related model, sharing similar properties, appeared at the same time [12]).
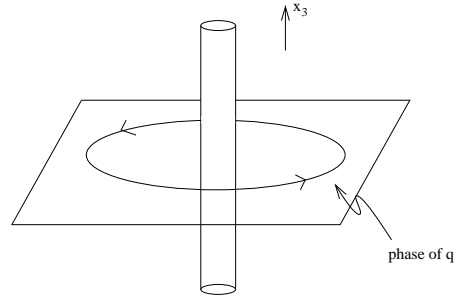
3.2 The Vortex Equations

To derive the vortex equations we once again perform the Bogomoln'yi completing the square trick (due, once again, to Bogomoln'yi [7]). We look for static strings in the x^3 direction, so make the ansatz $\partial_0 = \partial_3 = 0$ and $A_0 = A_3 = 0$. We also set $\phi = 0$. In fact ϕ will not play a role for the remainder of this lecture, although it will be resurrected

in the following lecture. The tension (energy per unit length) of the string is

$$T_{\text{vortex}} = \int dx^1 dx^2 \, \text{Tr} \, \left(\frac{1}{e^2} B_3^2 + \frac{e^2}{4} (\sum_{i=1}^{N} q_i q_i^\dagger - v^2 \, 1_N)^2 \right) + \sum_{i=1}^{N} |\mathcal{D}_1 q_i|^2 + |\mathcal{D}_2 q_i|^2$$

$$= \int dx^1 dx^2 \, \frac{1}{e^2} \text{Tr} \, \left(B_3 \mp \frac{e^2}{2} (\sum_{i=1}^{N} q_i q_i^\dagger - v^2 \, 1_N) \right)^2 + \sum_{i=1}^{N} |\mathcal{D}_1 q_i \mp i\mathcal{D}_2 q_i|^2$$

$$\mp v^2 \int dx^1 dx^2 \, \text{Tr} \, B_3 \tag{3.8}$$

To get from the first line to the second, we need to use the fact that $[D_1, D_2] = -iB_3$, to cancel the cross terms from the two squares. Using (3.7), we find that the tension of the charge $|k|$ vortex is bounded by

$$T_{\text{vortex}} \geq 2\pi v^2 \, |k| \tag{3.9}$$

In what follows we focus on vortex solutions with winding $k < 0$. (These are mapped into $k > 0$ vortices by a parity transformation, so there is no loss of generality). The inequality is then saturated for configurations obeying the vortex equations

$$B_3 = \frac{e^2}{2} (\sum_i q_i q_i^\dagger - v^2 \, 1_N) \quad , \quad \mathcal{D}_z q_i = 0 \tag{3.10}$$

where we've introduced the complex coordinate $z = x^1 + ix^2$ on the plane transverse to the vortex string, so $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. If we choose $N = 1$, then the Lagrangian (3.1) reduces to the abelian-Higgs model and, until recently, attention mostly focussed on this abelian variety of the equations (3.10). However, as we shall see below, when the vortex equations are non-abelian, so each side of the first equation (3.10) is an $N \times N$ matrix, they have a much more interesting structure.

Unlike monopoles and instantons, no analytic solution to the vortex equations is known. This is true even for a single $k = 1$ vortex in the $U(1)$ theory. There's nothing sinister about this. It's just that differential equations are hard and no one has decided to call the vortex solution a special function and give it a name! However, it's not difficult to plot the solution numerically and the profile of the fields is sketched below. The energy density is localized within a core of the vortex of size $L = 1/ev$, outside of which all fields return exponentially to their vacuum.

The simplest $k = 1$ vortex in the abelian $N = 1$ theory has just two collective coordinates, corresponding to its position on the z-plane. But what are the collective coordinates of a vortex in $U(N)$? We can use the same idea we saw in the instanton
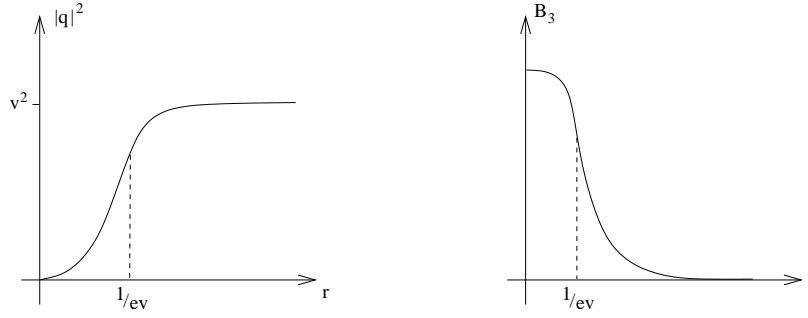
Figure 2: A sketch of the vortex profile.

lecture, and embed the abelian vortex — let's denote it $q^\star$ and $A_z^\star$ — in the $N \times N$ matrices of the non-abelian theory. We have

$$
A_z = \begin{pmatrix} A_z^\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \quad , \quad q = \begin{pmatrix} q^\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix} \tag{3.11}
$$

where the columns of the q matrix carry the color charge, while the rows carry the flavor charge. We have chosen the embedding above to lie in the upper left-hand corner but this isn't unique. We can rotate into other embeddings by acting with the $SU(N)_{\text{diag}}$ symmetry preserved in the vacuum. Dividing by the stabilizer, we find the internal moduli space of the single non-abelian vortex to be

$$
SU(N)_{\text{diag}}/S[U(N-1) \times U(1)] \cong \mathbb{CP}^{N-1} \tag{3.12}
$$

The appearance of $\mathbb{CP}^{N-1}$ as the internal space of the vortex is interesting: it tells us that the low-energy dynamics of a vortex string is the much studied quantum $\mathbb{CP}^{N-1}$ sigma model. We'll see the significance of this in the following lecture. For now, let's look more closely at the moduli of the vortices.

3.3 The Moduli Space

We've seen that a single vortex has $2N$ collective coordinates: 2 translations, and $2(N-1)$ internal modes, dictating the orientation of the vortex in color and flavor space. We denote the moduli space of charge k vortices in the $U(N)$ gauge theory as $\mathcal{V}_{k,N}$. We've learnt above that

$$
\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.13}
$$

What about higher k? An index theorem [14, 11] tells us that the number of collective coordinates is

$$\dim(\mathcal{V}_{k,N}) = 2kN \qquad (3.14)$$

Look familiar? Remember the result for k instantons in $U(N)$ that we found in lecture 1: $\dim(\mathcal{I}_{k,N}) = 4kN$. We'll see more of this similarity between instantons and vortices in the following.

As for previous solitons, the counting (3.14) has a natural interpretation: k parallel vortex strings may be placed at arbitrary positions, each carrying $2(N-1)$ independent orientational modes. Thinking physically in terms of forces between vortices, this is a consequence of tuning the coefficient $e^2/4$ in front of the D-term in (3.1) so that the mass of the gauge bosons equals the mass of the q scalars. If this coupling is turned up, the scalar mass increases and so mediates a force with shorter range than the gauge bosons, causing the vortices to repel. (Recall the general rule: spin 0 particles give rise to attractive forces; spin 1 repulsive). This is a type II non-abelian superconductor. If the coupling decreases, the mass of the scalar decreases and the vortices attract. This is a non-abelian type I superconductor. In the following, we keep with the critically coupled case (3.1) for which the first order equations (3.10) yield solutions with vortices at arbitrary position.

3.3.1 The Moduli Space Metric

There is again a natural metric on $\mathcal{V}_{k,N}$ arising from taking the overlap of zero modes. These zero modes must solve the linearized vortex equations together with a suitable background gauge fixing condition. The linearized vortex equations read

$$\mathcal{D}_z \delta A_{\bar{z}} - \mathcal{D}_{\bar{z}} \delta A_z = \frac{ie^2}{4}(\delta q\, q^\dagger + q\, \delta q^\dagger) \quad \text{and} \quad \mathcal{D}_z \delta q = i\delta A_z q \qquad (3.15)$$

where q is to be viewed as an $N \times N$ matrix in these equations. The gauge fixing condition is

$$\mathcal{D}_z \delta A_{\bar{z}} + \mathcal{D}_{\bar{z}} \delta A_z = -\frac{ie^2}{4}(\delta q\, q^\dagger - q\, \delta q^\dagger) \qquad (3.16)$$

which combines with the first equation in (3.15) to give

$$\mathcal{D}_{\bar{z}} \delta A_z = -\frac{ie^2}{4}\delta q\, q^\dagger \qquad (3.17)$$

Then, from the index theorem, we know that there are $2kN$ zero modes $(\delta_\alpha A_z, \delta_\alpha q)$, $\alpha, \beta = 1, \ldots, 2kN$ solving these equations, providing a metric on $\mathcal{V}_{k,N}$ defined by

$$g_{\alpha\beta} = \text{Tr} \int dx^1 dx^2 \; \frac{1}{e^2}\delta_\alpha A_a \delta_\beta A_{\bar{z}} + \frac{1}{2}\delta_\alpha q \delta_\beta q^\dagger \; + \; \text{h.c.} \qquad (3.18)$$

The metric has the following properties [15, 16]

- The metric is Kähler. This follows from similar arguments to those given for hyperKählerity of the instanton moduli space, the complex structure now descending from that on the plane $\mathbf{R}^2$, together with the obvious complex structure on q.

- The metric is smooth. It has no singularities as the vortices approach each other. Strictly speaking this statement has been proven only for abelian vortices. For non-abelian vortices, we shall show this using branes in the following section.

- The metric inherits a $U(1) \times SU(N)$ holomorphic isometry from the rotational and internal symmetry of the Lagrangian.

- The metric is unknown for $k \geq 2$. The leading order, exponentially suppressed, corrections to the flat metric were computed recently [17].

3.3.2 Examples of Vortex Moduli Spaces

A Single $U(N)$ Vortex

We've already seen above that the moduli space for a single $k = 1$ vortex in $U(N)$ is

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.19}$$

where the isometry group $SU(N)$ ensures that $\mathbb{CP}^{N-1}$ is endowed with the round, Fubini-Study metric. The only question remaining is the size, or Kähler class, of the $\mathbb{CP}^{N-1}$. This can be computed either from a D-brane construction [11] or, more conventionally, from the overlap of zero modes [18]. We'll see the former in the following section. Here let's sketch the latter. The orientational zero modes of the vortex take the form

$$\delta A_z = \mathcal{D}_z \Omega \quad , \quad \delta q = i(\Omega q - q \Omega_0) \tag{3.20}$$

where the gauge transformation asymptotes to $\Omega \to \Omega_0$, and Ω_0 is the flavor transformation. The gauge fixing condition requires

$$\mathcal{D}^2 \Omega = \frac{e^2}{2}\{\Omega, qq^\dagger\} - 2qq^\dagger \Omega_0 \tag{3.21}$$

By explicitly computing the overlap of these zero modes, it can be shown that the size of the $\mathbb{CP}^{N-1}$ is

$$r = \frac{4\pi}{e^2} \tag{3.22}$$

This important equation will play a crucial role in the correspondence between 2d sigma models and 4d gauge theories that we'll meet in the following lecture.

Two $U(1)$ Vortices

The moduli space of two vortices in a $U(1)$ gauge theory is topologically

$$\mathcal{V}_{k=2,N=1} \cong \mathbf{C} \times \mathbf{C}/\mathbf{Z}_2 \qquad (3.23)$$

where the $\mathbf{Z}_2$ reflects the fact that the two solitons are indistinguishable. Note that the notation we used above actually describes more than the topology of the manifold because, topologically, $\mathbf{C}^k/\mathbf{Z}_k \cong \mathbf{C}^k$ (as any polynomial will tell you). So when I write $\mathbf{C}/\mathbf{Z}_2$ in (3.23), I mean that asymptotically the space is endowed with the flat metric on $\mathbf{C}/\mathbf{Z}_2$. Of course, this can't be true closer to the origin since we know the vortex moduli space is complete. The cone must be smooth at the tip, as shown in figure 3. The metric on the cone has been computed numerically [19], no analytic form is known. The deviations from the flat, singular, metric on the cone are exponentially suppressed and parameterized by the size of the vortex $L \sim 1/ev$.
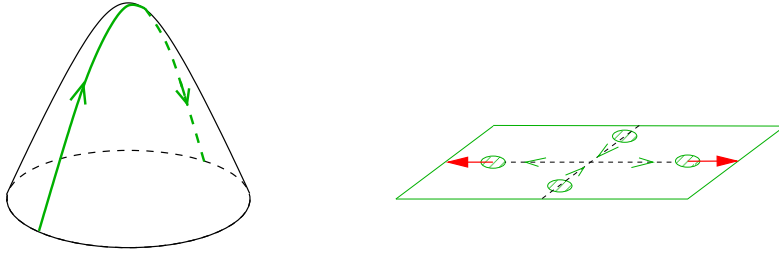

Figure 3: Right-angle scattering from the moduli space of two vortices.

Even without the exact form of the metric, we learn something very important about vortices. Consider two vortices colliding head on. This corresponds to the trajectory in moduli space that goes up and over the tip of the cone, as shown in the figure. What does this correspond to in real space? One might think that the vortices collide and rebound. But that's wrong: it would correspond to the trajectory going to the tip of the cone, and returning down the same side. Instead, the trajectory corresponds to vortices scattering at right angles [20]. The key point is that the $\mathbf{Z}_2$ action in (3.23), arising because the vortices are identical, means that the single valued coordinate on the moduli space is z^2 rather than z, the separation between the vortices. The collision sends $z^2 \rightarrow -z^2$ or $z \rightarrow iz$. This result doesn't depend on the details of the metric on the vortex moduli space, but follows simply from the fact that, near the origin, the space is smooth. Right-angle scattering of this type is characteristic of soliton collisions, occurring also for magnetic monopoles.

For $k \geq 3$ $U(1)$ vortices, the moduli space is topologically and asymptotically $\mathbf{C}^k/\mathbf{Z}_k$. The leading order exponential corrections to the flat metric on this space are known, although the full metric is not [17].

3.4 Brane Construction

For both instantons and monopoles, it was fruitful to examine the solitons from the perspective of D-branes. This allowed us to re-derive the ADHM and Nahm constructions respectively. What about for vortices? Here we present a D-brane constuction of vortices [11] that will reveal interesting information about the moduli space of solutions although, ultimately, won't be as powerful as the ADHM and Nahm constructions described in previous sections.

We use the brane set-ups of Hanany and Witten [21], consisting of D-branes suspended between a pair of NS5-branes. We work in type IIA string theory, and build the $d = 3+1$, $U(N)$ gauge theory[1] with $\mathcal{N} = 2$ supersymmetry. The D-brane set-up is shown in figure 4, and consists of N D4-branes with worldvolume 01236, stretched between two NS5-branes, each with worldvolume 012345, and separated in the x^6 direction. The gauge coupling e^2 is determined by the separation between the NS5-branes,

$$\frac{1}{e^2} = \frac{\Delta x^6 \, l_s}{2g_s} \tag{3.24}$$

where l_s is the string length, and g_s the string coupling.

Figure 4:

The D4-branes may slide up and down between the NS5-branes in the x^4 and x^5 direction. This corresponds to turning on a vev for the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet. Since we consider only a real adjoint scalar ϕ in our theory, we have

$$\phi_a = \left.\frac{x^4}{l_s^2}\right|_{D4_a} \tag{3.25}$$

and we'll take all D4-branes to lie coincident in the x^5 direction.

The hypermultiplets arise in the form of N D6-branes with worldvolume 0123789. The positions of the D6-branes in the $x^4 + ix^5$ directions will correspond to complex masses for the hypermultiplets. We shall consider these in the following section, but for now we set all D6-branes to lie at the origin of the x^4 and x^5 plane.
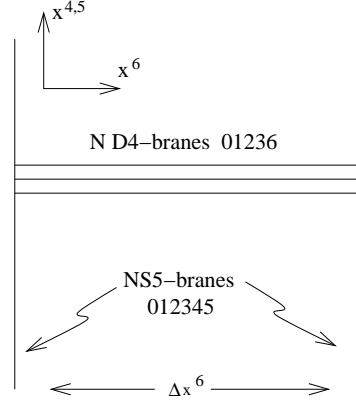
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[1]In fact, for four-dimensional theories the overall $U(1)$ decouples in the brane set-up, and we have only $SU(N)$ gauge theory [22]. This doesn't affect our study of the vortex moduli space; if you're bothered by this, simply T-dualize the problem to type IIB where you can study vortices in $d = 2+1$ dimensions.

We also need to turn on the FI parameter v^2. This is achieved by taking the right-hand NS5-brane and pulling it out of the page in the x^9 direction. In order to remain in the ground state, the D4-branes are not allowed to tilt into the x^9 direction: this would break supersymmetry and increase their length, reflecting a corresponding increase in the ground state energy of the theory. Instead, they must split on the D6-branes. Something known as the S-rule [21, 23] tells us that only one D4-brane can end on a given D6-brane while preserving supersymmetry, ensuring that we need at least N D6-branes to find a zero-energy ground state. The final configuration is drawn in the figure 6, with the field theory dictionary given by

Figure 5:

$$v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3} \tag{3.26}$$

Now we've built our theory, we can look to find the vortices. We expect them to appear as other D-branes in the configuration. There is a unique BPS D-brane with the correct mass: it is a D2-brane, lying coincident with the D6-branes, with worldvolume 039, as shown in figure 6 [24]. The x^3 direction here is the direction of the vortex string.

The problem is: what is the worldvolume theory on the D2-branes. It's hard to read off the theory directly because of the boundary conditions where the D2-branes end on the D4-branes. But, already by inspection, we might expect that it's related to the Dp-D$(p-4)$ system described in Lecture 1 in the context of instantons. To make progress we

Figure 6:

play some brane games. Move the D6-branes to the right. As they pass the NS5-brane, the Hanany-Witten transition occurs and the right-hand D4-branes disappear [21]. We get the configuration shown in figure 7.

Let's keep the D6-branes moving. Off to infinity. Finally, we rotate our perspective a little, viewing the D-branes from a different angle, shown in figure 8. This is our final D-brane configuration and we can now read off the dynamics.

We want to determine the theory on the D2-branes in figure 8. Let's start with the easier problem in figure 9. Here the D4-branes extend to infinity in both $x^6 \to \pm\infty$ directions, and the D2-branes end on the other NS5. The theory on the D2-branes
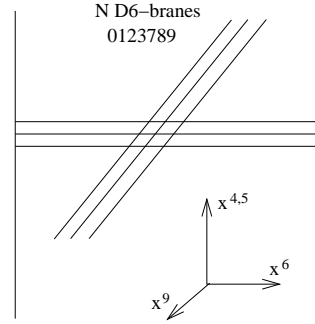
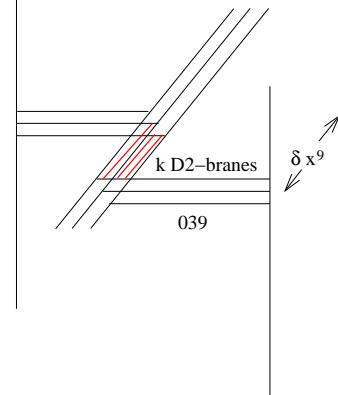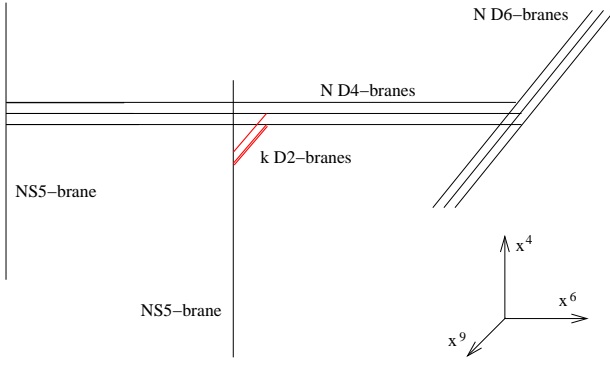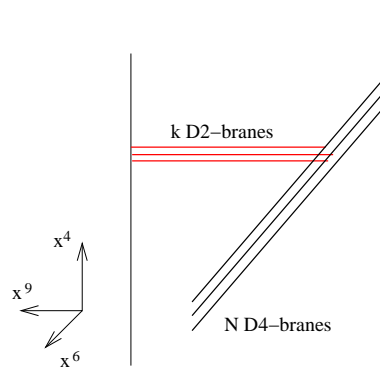**Figure 7:** Moving the D6-branes

Figure 8: Rotating our viewpoint

is simple to determine: it is a $U(k)$ gauge theory with 4 real adjoint scalars, or two complex scalars

$$\sigma = X^4 + iX^5 \quad , \quad Z = X^1 + iX^2 \tag{3.27}$$

which combine to give the $\mathcal{N} = (4,4)$ theory in $d = 1 + 1$. The D4-branes contribute hypermultiplets $(\psi_a, \tilde{\psi}_a)$ with $a = 1, \ldots, N$. These hypermultiplets get a mass only when the D2-branes and D4-branes are separated in the X^4 and X^5 directions. This means we have a coupling like

Figure 9:

$$\sum_{a=1}^{N} \psi_a^\dagger \{\sigma^\dagger, \sigma\} \psi_a + \tilde{\psi}_a \{\sigma^\dagger, \sigma\} \tilde{\psi}_a^\dagger \tag{3.28}$$

But there is no such coupling between the hypermultiplets and Z. The coupling (3.28) breaks supersymmetry to $\mathcal{N} = (2,2)$. So we now understand the D2-brane theory of figure 9. However, the D2-brane theory that we're really interested in, shown in figure 8, differs from this in two ways

- The right-hand NS5-brane is moved out of the page. But we already saw in the manoeuvres around figure 6 that this induces a FI parameter on brane theory. Except this this time the FI parameter is for the D2-brane theory. It's given by

$$r = \frac{\Delta x^6}{2\pi g_s l_s} = \frac{4\pi}{e^2} \tag{3.29}$$

- We only have half of the D4-branes, not all of them. If a full D4-brane gives rise to a hypermultiplet, one might guess that half a D4-brane should give rise to half a hypermultiplet, otherwise known as a chiral multiplet. Although the argument is a little glib, it turns out that this is the correct answer [25].
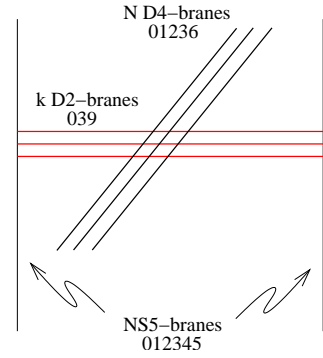
We end up with the gauge theory in $d = 1 + 1$ dimensions with $\mathcal{N} = (2, 2)$ supersymmetry

$$U(k) \text{ Gauge Theory } + \text{ Adjoint Chiral Multiplet } Z$$
$$+ \; N \text{ Fundamental Chiral Multiplets } \psi_a$$

This theory has a FI parameter $r = 4\pi/e^2$. Now this should be looking very familiar — it's very similar to the instanton theory we described in Lecture 1. We'll return to this shortly. For now let's keep examining our vortex theory. The potential for the various scalars is dictated by supersymmetry and is given by

$$V = \frac{1}{g^2} \text{Tr} \, |[\sigma, \sigma^\dagger]|^2 + \text{Tr} \, |[\sigma, Z]|^2 + \text{Tr} \, |[\sigma, Z^\dagger]|^2 + \sum_{a=1}^{N} \psi_a^\dagger \sigma^\dagger \sigma \psi_a$$
$$+ \frac{g^2}{2} \text{Tr} \left(\sum_a \psi_a \psi_a^\dagger + [Z, Z^\dagger] - r \, 1_k \right)^2 \tag{3.30}$$

Here g^2 is an auxiliary gauge coupling which we take to infinity $g^2 \to \infty$ to restrict us to the Higgs branch, the vacuum moduli space defined by

$$\mathcal{M}_{\text{Higgs}} \cong \{\sigma = 0, V = 0\}/U(k) \tag{3.31}$$

Counting the various degrees of freedom, the Higgs branch has real dimension $2kN$. From the analogy with the instanton case, it is natural to conjecture that this is the vortex moduli space [11]

$$\mathcal{V}_{k,N} \cong \mathcal{M}_{\text{Higgs}} \tag{3.32}$$

While the ADHM construction has a field theoretic underpinning, I know of no field theory derivation of the above result for vortices. So what evidence do we have that the Higgs branch indeed coincides with the vortex moduli space? Because of the FI parameter, $\mathcal{M}_{\text{Higgs}}$ is a smooth manifold, as is $\mathcal{V}_{k,N}$ and, obviously the dimensions work out. Both spaces have a $SU(N) \times U(1)$ isometry which, in the above construction, act upon ψ and Z respectively. Finally, in all cases we can check, the two spaces agree (as, indeed, do their Kähler classes). Let's look at some examples.

3.4.1 Examples of Vortex Moduli Spaces Revisited

One Vortex in $U(N)$

The gauge theory for a single $k = 1$ vortex in $U(N)$ is a $U(1)$ gauge theory. The adjoint scalar Z decouples, parameterizing the complex plane $\mathbf{C}$, leaving us with the N charged

scalars satisfying

$$\sum_{a=1}^{N} |\psi_a|^2 = r \tag{3.33}$$

modulo the $U(1)$ action $\psi_a \to e^{i\alpha}\psi_a$. This gives us the moduli space

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{3.34}$$

where the $\mathbb{CP}^{N-1}$ has the correct Kähler class $r = 4\pi/e^2$ in agreement with (3.22). The metric on $\mathbb{CP}^{N-1}$ is, again, the round Fubini-Study metric.

k Vortices in $U(1)$

The Higgs branch corresponding to the k vortex moduli space is

$$\{\psi\psi^\dagger + [Z, Z^\dagger] = r\,\mathbf{1}_k\}/U(k) \tag{3.35}$$

which is asymptotic to the cone $\mathbf{C}^k/\mathbf{Z}_k$, with the singularities resolved. This is in agreement with the vortex moduli space. *However*, the metric on $\mathcal{M}_{\text{Higgs}}$ differs by power law corrections from the flat metric on the orbifold $\mathbf{C}^k/\mathbf{Z}_k$. But, as we've discussed, $\mathcal{V}_{k,N}$ differs from the flat metric by exponential corrections.

More recently, the moduli space of two vortices in $U(N)$ was studied in some detail and shown to possess interesting and non-trivial topology [26], with certain expected features of $\mathcal{V}_{2,N}$ reproduced by the Higgs branch.

In summary, it is conjectured that the vortex moduli space $\mathcal{V}_{k,N}$ is isomorphic to the Higgs branch (3.34). But, except for the case $k = 1$ where the metric is determined by the isometry, the metrics do not agree. A direct field theory proof of this correspondence remains to be found.

3.4.2 The Relationship to Instantons

As we've mentioned a few times, the vortex theory bears a striking resemblance to the ADHM instanton theory we met in Lecture 1. In fact, the gauge theoretic construction of vortex moduli space $\mathcal{V}_{k,N}$ involves exactly half the fields of the ADHM construction. Or, put another way, the vortex moduli space is half of the instanton moduli space. We can state this more precisely: $\mathcal{V}_{k,N}$ is a complex, middle dimensional submanifold of $\mathcal{I}_{k,N}$. It can be defined by looking at the action of the isometry rotating the instantons in the $x^3 - x^4$ plane. Denote the corresponding Killing vector as h. Then

$$\mathcal{V}_{k,N} \cong \mathcal{I}_{k,N}|_{h=0} \tag{3.36}$$

where $\mathcal{I}_{k,N}$ is the resolved instanton moduli space with non-commutativity parameter $\theta_{\mu\nu} = r\bar{\eta}_{\mu\nu}^3$. We'll see a physical reason for this relationship shortly.

An open question: The ADHM construction is constructive. As we have seen, it allows us to build solutions to $F = {}^\star F$ from the variables of the Higgs branch. Does a similar construction exist for vortices?

Relationships between the instanton and vortex equations have been noted in the past. In particular, a twisted reduction of instantons in $SU(2)$ Yang-Mills on $\mathbf{R}^2 \times \mathbf{S}^2$ gives rise to the $U(1)$ vortex equations [27]. While this relationship appears to share several characteristics to the correspondence described above, it differs in many important details. It don't understand the relationship between the two approaches.

3.5 Adding Flavors

Let's now look at vortices in a $U(N_c)$ gauge theory with $N_f \geq N_c$ flavors. Note that we've added subscripts to denote color and flavor. In theories with $N_c = 1$ and $N_f > 1$, these were called semi-local vortices [28, 29, 30, 31]. The name derives from the fact the theory has both a gauge (local) group and a flavor (global) group. But for us, it's not a great name as all our theories have both types of symmetries, but it's only when $N_f > N_c$ that the extra properties of "semi-local" vortices become apparent.

The Lagrangian (3.1) remains but, unlike before, the theory no longer has a mass gap in vacuum. Instead there are N_c^2 massive scalar fields and scalars, and $2N_c(N_f - N_c)$ massless scalars. At low-energies, the theory reduces to a σ-model on the Higgs branch of the gauge theory (3.1),

$$\mathcal{M}_{\text{Higgs}} \cong \{\sum_{i=1}^{N_f} q_i q_i^\dagger = v^2 \, 1_{N_c}\}/U(N_c) \cong G(N_c, N_f) \tag{3.37}$$

When we have an abelian $N_c = 1$ theory, this Higgs branch is the projective space $G(1, N_f) \cong \mathbb{CP}^{N_f - 1}$. For non-abelian theories, the Higgs branch is the Grassmannian $G(N_c, N_f)$, the space of $\mathbf{C}^{N_c}$ planes in $\mathbf{C}^{N_f}$. In a given vacuum, the symmetry breaking pattern is $U(N_c) \times SU(N_f) \to S[U(N_c) \times U(N_f - N_c)]$.

The first order vortex equations (3.10) still give solutions to the full Lagrangian, now with the flavor index running over values $i = 1 \ldots, N_f$. Let's denote the corresponding vortex moduli space as $\hat{\mathcal{V}}_{k,N_c,N_f}$, so our previous notation becomes $\mathcal{V}_{k,N} \cong \hat{\mathcal{V}}_{k,N,N}$. The index theorem now tells us the dimension of the vortex moduli space

$$\dim(\hat{\mathcal{V}}_{k,N_c,N_f}) = 2kN_f \tag{3.38}$$

The dimension depends only on the number of flavors, and the semi-local vortices inherit new modes. These modes are related to scaling modes of the vortex — the size of the vortex becomes a parameter, just as it was for instantons [32].

These vortices arising in the theory with extra flavors are related to other solitons, known as a sigma-model lumps. (These solitons have other names, depending on the context, sometimes referred to as "textures", "Skyrmions" or, in the context of string theory, "worldsheet instantons"). Let's see how this works. At low-energies (or, equivalently, in the strong coupling limit $e^2 \to \infty$) our gauge theory flows to the sigma-model on the Higgs branch $\mathcal{M}_{\text{Higgs}} \cong G(N_c, N_f)$. In this limit our vortices descend to lumps, objects which gain their topological support once we compactify the $(x^1 - x^2)$-plane at infinity, and wrap this sphere around $\mathcal{M}_{\text{Higgs}} \cong G(N_c, N_f)$ [33, 34]

$$\Pi_2(G(N_c, N_f)) \cong \mathbf{Z} \tag{3.39}$$

When $N_f = N_c$ there is no Higgs branch, the vortices have size $L = 1/ev$ and become singular as $e^2 \to \infty$. In contrast, when $N_f > N_c$, the vortices may have arbitrary size and survive the strong coupling limit. However, while the vortex moduli space is smooth, the lump moduli space has singularities, akin to the small instanton singularities we saw in Lecture 1. We see that the gauge coupling $1/e^2$ plays the same role for lumps as θ plays for Yang-Mills instantons.

The brane construction for these vortices is much like the previous section - we just need more $D6$ branes. By performing the same series of manoeuvres, we can deduce the worldvolume theory. It is again a $d = 1+1$ dimensional, $\mathcal{N} = (2,2)$ theory with

$U(k)$ Gauge Theory $+$ Adjoint Chiral Multiplet Z

$+$ N_c Fundamental Chiral Multiplets ψ_a

$+$ $(N_f - N_c)$ Anti-Fundamental Chiral Multiplets $\tilde{\psi}_a$

Once more, the FI parameter is $r = 4\pi/e^2$. The D-term constraint of this theory is

$$\sum_{a=1}^{N_c} \psi_a \psi_a^\dagger - \sum_{b=1}^{N_f - N_c} \tilde{\psi}_b^\dagger \tilde{\psi}_b + [Z, Z^\dagger] = r \, 1_k \tag{3.40}$$

A few comments

- Unlike the moduli space $\mathcal{V}_{k,N}$, the presence of the $\tilde{\psi}$ means that this space doesn't collapse as we send $r \to 0$. Instead, in this limit it develops singularities at $\psi = \tilde{\psi} = 0$ where the $U(k)$ gauge group doesn't act freely. This is the manifestation of the discussion above.

- The metric inherited from the D-term (3.40) again doesn't coincide with the metric on the vortex moduli space $\hat{\mathcal{V}}_{k,N_c,N_f}$. In fact, here the discrepancy is

more pronounced, since the metric on $\hat{\mathcal{V}}_{k,N_c,N_f}$ has non-normalizable modes: the directions in moduli space corresponding to the scaling the solution are suffer an infra-red logarithmic divergence [35, 32]. The vortex theory arising from branes doesn't capture this.

3.5.1 Non-Commutative Vortices

As for instantons, we can consider vortices on the non-commutative plane

$$[x^1, x^2] = i\vartheta \tag{3.41}$$

These objects were first studied in [36]. How does this affect the moduli space? In the ADHM construction for instantons, we saw that non-commutivity added a FI parameter to the D-term constraints. But, for vortices, we already have a FI parameter: $r = 4\pi/e^2$. It's not hard to show using D-branes [11], that the effect of non-commutivity is to deform,

$$r = \frac{4\pi}{e^2} + 2\pi v^2 \vartheta \tag{3.42}$$

This has some interesting consequences. Note that for $N_f = N_c$, there is a critical FI parameter $\vartheta_c = -v^2/e^2$ for which $r = 0$. At this point the vortex moduli space becomes singular. For $\vartheta < \vartheta_c$, no solutions to the D-term equations exist. Indeed, it can be shown that in this region, no solutions to the vortex equations exist either [37]. We see that the Higgs branch correctly captures the physics of the vortices.

For $N_f > N_c$, the Higgs branch makes an interesting prediction: the vortex moduli space should undergo a topology changing transition as $\vartheta \rightarrow \vartheta_c$. For example, in the case of a single $k = 1$ vortex in $U(2)$ with $N_f = 4$, this is the well-known flop transition of the conifold. To my knowledge, no one has confirmed this behavior of the vortex moduli space from field theory. Nor has anyone found a use for it!

3.6 What Became Of.......

Let's now look at what became of the other solitons we studied in the past two lectures.

3.6.1 Monopoles

Well, we've set $\phi = 0$ throughout this lecture and, as we saw, the monopoles live on the vev of ϕ. So we shouldn't be surprised if they don't exist in our theory (3.1). We'll see them reappear in the following section.

3.6.2 Instantons

These are more interesting. Firstly the vev $q \neq 0$ breaks conformal invariance, causing the instantons to collapse. This is the same behavior that we saw in Section 2.6. But recall that in the middle of the vortex string, $q \to 0$. So maybe it's possible for the instanton to live inside the vortex string, where the non-abelian gauge symmetry is restored. To see that this can indeed occur, we can look at the worldsheet of the vortex string. As we've seen, the low-energy dynamics for a single string is

$$U(1) \text{ with N charged chiral multiplets and FI parameter } r = 4\pi/e^2$$

But this falls into the class of theories we discussed in section 3.5. So if the worldsheet is Euclidean, the theory on the vortex string itself admits a vortex solution: a vortex in a vortex. The action of this vortex is [38]

$$S_{\text{vortex in vortex}} = 2\pi r = \frac{8\pi^2}{e^2} = S_{\text{inst}} \tag{3.43}$$

which is precisely the action of the Yang-Mills instanton. Such a vortex has $2N$ zero modes which include scaling modes but, as we mentioned previously, not all are normalizable.

There is also a 4d story for these instantons buried in the vortex string. This arises by completing the square in the Lagrangian in a different way to (3.8). We still set $\phi = 0$, but now allow for all fields to vary in all four dimensions [38]. We write $z = x^1 + ix^2$ and $w = x^3 - ix^4$,

$$
\begin{aligned}
S &= \int d^4x \; \frac{1}{2e^2}\text{Tr } F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 + \frac{e^2}{4}\text{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \mathbf{1}_{N_c})^2 \\
&= \int d^4x \; \frac{1}{2e^2}\text{Tr } \left(F_{12} - F_{34} - \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \mathbf{1}_{N_c}) \right)^2 \\
&\quad + \sum_{i=1}^{N_f} |\mathcal{D}_z q_i|^2 + |\mathcal{D}_w q_i|^2 + \frac{1}{e^2}\text{Tr}\left((F_{14} - F_{23})^2 + (F_{13} + F_{24})^2 \right) \\
&\quad + \frac{1}{e^2}\text{Tr } F_{\mu\nu}{}^\star F^{\mu\nu} + F_{12}v^2 + F_{34}v^2 \\
&\geq \int d^4x \; \frac{1}{e^2}\text{Tr } F_{\mu\nu}{}^\star F^{\mu\nu} + \text{Tr}\left(F_{12}v^2 + F_{34}v^2 \right) \tag{3.44}
\end{aligned}
$$

The last line includes three topological charges, corresponding to instantons, vortex strings in the $(x^1 - x^2)$ plane, and further vortex strings in the $(x^3 - x^4)$ plane. The

Bogomoln'yi equations describing these composite solutions are

$$F_{14} = F_{23} \quad , \quad F_{13} = F_{24} \quad , \quad F_{12} - F_{34} = \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2 \mathbf{1}_{N_c}) \quad , \quad \mathcal{D}_z q_i = \mathcal{D}_w q_i = 0$$

It is not known if solutions exist, but the previous argument strongly suggests that there should be solutions describing an instanton trapped inside a vortex string. Some properties of this configuration were studied in [39].

The observation that a vortex in the vortex string is a Yang-Mills instanton gives some rationale to the fact that $\mathcal{V}_{k,N} \subset \mathcal{I}_{k,N}$.

3.7 Fermi Zero Modes

In this section, I'd like to describe an important feature of fermionic zero modes on the vortex string: they are chiral. This means that a Weyl fermion in four dimensions will give rise to a purely left-moving (or right-moving) mode on the (anti-) vortex worldsheet. In fact, a similar behavior occurs for instantons and monopoles, but since this is the first lecture where the solitons are string-like in four-dimensions, it makes sense to discuss this phenomenon here.

The exact nature of the fermionic zero modes depends on the fermion content in four dimensions. Let's stick with the supersymmetric generalization of the Lagrangian (3.1). Then we have the gaugino λ, an adjoint valued Weyl fermion which is the superpartner of the gauge field. We also have fermions in the fundamental representation, χ_i with $i = 1, \ldots, N$, which are the superpartners of the scalars q_i. These two fermions mix through Yukawa couplings of the form $q_i^\dagger \lambda \chi_i$, and the Dirac equations read

$$-i \, \slashed{\bar{D}} \lambda + i\sqrt{2} \sum_{i=1}^{N} q_i \bar{\chi}_i = 0 \quad \text{and} \quad -i \, \slashed{D} \bar{\chi}_i - i\sqrt{2} q_i^\dagger \lambda = 0 \tag{3.45}$$

where the Dirac operators take the form,

$$\slashed{D} \equiv \sigma^\mu \mathcal{D}_\mu = \begin{pmatrix} \mathcal{D}_+ & \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} & \mathcal{D}_- \end{pmatrix} \quad \text{and} \quad \slashed{\bar{D}} \equiv \bar{\sigma}^\mu \mathcal{D}_\mu = \begin{pmatrix} \mathcal{D}_- & -\mathcal{D}_z \\ -\mathcal{D}_{\bar{z}} & \mathcal{D}_+ \end{pmatrix} \tag{3.46}$$

which, as we can see, nicely split into $\mathcal{D}_\pm = \mathcal{D}_0 \pm \mathcal{D}_3$ and $\mathcal{D}_z = \mathcal{D}_1 - i\mathcal{D}_2$ and $\mathcal{D}_{\bar{z}} = \mathcal{D}_1 + i\mathcal{D}_2$. The bosonic fields in (3.45) are evaluated on the vortex solution which, crucially, includes $\mathcal{D}_z q_i = 0$ for the vortex (or $\mathcal{D}_{\bar{z}} q_i = 0$ for the anti-vortex). We see the importance of this if we take the first equation in (3.45) and hit it with $\slashed{D}$, while hitting the second equation with $\slashed{\bar{D}}$. In each equation terms of the form $\mathcal{D}_z q_i$ will

appear, and subsequently vanish as we evaluate them on the vortex background. Let's do the calculation. We split up the spinors into their components λ_α and $(\chi_\alpha)_i$ with $\alpha = 1, 2$ and, for now, look for zero modes that don't propagate along the string, so $\partial_+ = \partial_- = 0$. Then the Dirac equations in component form become

$$(-\mathcal{D}_z \mathcal{D}_{\bar{z}} + 2q_i q_i^\dagger)\lambda_1 = 0 \quad \text{and} \quad (-\mathcal{D}_{\bar{z}} \mathcal{D}_z + 2q_i q_i^\dagger)\lambda_2 - \sqrt{2}(\mathcal{D}_{\bar{z}} q_i)\bar{\chi}_{1i} = 0$$
$$(-\mathcal{D}_{\bar{z}} \mathcal{D}_z \delta^j{}_i + 2q_i^\dagger q_j)\bar{\chi}_{2j} = 0 \quad \text{and} \quad (-\mathcal{D}_z \mathcal{D}_{\bar{z}} \delta^j{}_i + 2q_i q_j^\dagger)\bar{\chi}_{1j} - \sqrt{2}(\mathcal{D}_z q_i^\dagger)\lambda_2 = 0$$

The key point is that the operators appearing in the first column are positive definite, ensuring that λ_1 and χ_{2i} have no zero modes. In contrast, the equations for λ_2 and $\bar{\chi}_{1i}$ do have zero modes, guaranteed by the index. We therefore know that any zero modes of the vortex are of the form,

$$\lambda = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \quad \text{and} \quad \bar{\chi}_i = \begin{pmatrix} \bar{\chi}_i \\ 0 \end{pmatrix} \tag{3.47}$$

If we repeat the analysis for the anti-vortex, we find that the other components turn on. To see the relationship to the chirality on the worldsheet, we now allow the zero modes to propagate along the string, so that $\lambda = \lambda(x^0, x^3)$ and $\bar{\chi}_i = \bar{\chi}_i(x^0, x^3)$. Plugging this ansatz back into the Dirac equation, now taking into account the derivatives $\mathcal{D}_\pm$ in (3.46), we find the equations of motion

$$\partial_+ \lambda = 0 \quad \text{and} \quad \partial_+ \bar{\chi}_i = 0 \tag{3.48}$$

Or, in other words, $\lambda = \lambda(x_-)$ and $\bar{\chi} = \bar{\chi}(x_-)$: both are right movers.

In fact, the four-dimensional theory with only fundamental fermions χ_i is anomalous. Happily, so is the $\mathbb{CP}^{N-1}$ theory on the string with only right-moving fermions, suffering from the sigma-model anomaly [40]. To rectify this, one may add four dimensional Weyl fermions $\tilde{\chi}_i$ in the anti-fundamental representation, which provide left movers on the worldsheet. If the four-dimensional theory has $\mathcal{N} = 2$ supersymmetry, the worldsheet theory preserves $\mathcal{N} = (2, 2)$ [41]. Alternatively, one may work with a chiral, non-anomalous $\mathcal{N} = 1$ theory in four-dimensions, resulting in a chiral non-anomalous $\mathcal{N} = (0, 2)$ theory on the worldsheet.

3.8 Applications

Let's now turn to discussion of applications of vortices in various field theoretic contexts. We review some of the roles vortices play as finite action, instanton-like, objects in two dimensions, as particles in three dimensions, and as strings in four dimensions.

3.8.1 Vortices and Mirror Symmetry

Perhaps the most important application of vortices in string theory is in the context of the $d = 1+1$ dimensional theory on the string itself. You might protest that the string worldsheet theory doesn't involve a gauge field, so why would it contain vortices?! The trick, as described by Witten [33], is to view sigma-models in terms of an auxiliary gauge theory known as a *gauged linear sigma model*. We've already met this trick several times in these lectures: the sigma-model target space is the Higgs branch of the gauge theory. Witten showed how to construct gauge theories that have compact Calabi-Yau manifolds as their Higgs branch.

In $d = 1+1$ dimensions, vortices are finite action solutions to the Euclidean equations of motion. In other words, they play the role of instantons in the theory. As we explained Section 1.5 above, the vortices are related to worldsheet instantons wrapping the 2-cycles of the Calabi-Yau Higgs branch. It turns out that it is much easier to deal with vortices than directly with worldsheet instantons (essentially because their moduli space is free from singularities). Indeed, in a beautiful paper, Morrison and Plesser succeeded in summing the contribution of all vortices in the topological A-model on certain Calabi-Yau manifolds, showing that it agreed with the classical prepotential derived from the B-model on the mirror Calabi-Yau [42].

More recently, Hori and Vafa used vortices to give a proof of $\mathcal{N} = (2,2)$ mirror symmetry for all Calabi-Yau which can be realized as complete intersections in toric varieties [43]. Hori and Vafa work with dual variables, performing the so-called Rocek-Verlinde transformation to twisted chiral superfields [44]. They show that vortices contribute to a two fermi correlation function which, in terms of these dual variables, is cooked up by a superpotential. This superpotential then captures the relevant quantum information about the original theory. Similar methods can be used in $\mathcal{N} = (4,4)$ theories to derive the T-duality between NS5-branes and ALE spaces [45, 46, 47, 48], with the instantons providing the necessary ingredient to break translational symmetry after T-duality, leading to localized, rather than smeared, NS5-branes.

3.8.2 Swapping Vortices and Electrons

In lecture 2, we saw that it was possible to rephrase four-dimensional field theories, treating the monopoles as elementary particles instead of solitons. This trick, called electric-magnetic duality, gives key insight into the strong coupling behavior of four-dimensional field theories. In three dimensions, vortices are particle like objects and one can ask the same question: is it possible to rewrite a quantum field theory, treating the vortices as fundamental degrees of freedom?

The answer is yes. In fact, condensed matter theorists have been using this trick for a number of years (see for example [49]). Things can be put on a much more precise footing in the supersymmetric context, with the first examples given by Intriligator and Seiberg [50]. They called this phenomenon "mirror symmetry" in three dimensions as it had some connection to the mirror symmetry of Calabi-Yau manifolds described above.

Let's describe the basic idea. Following Intriligator and Seiberg, we'll work with a theory with eight supercharges (which is $\mathcal{N} = 4$ supersymmetry in three dimensions). Each gauge field comes with three real scalars and four Majorana fermions. The charged matter, which we'll refer to as "electrons", lives in a hypermultiplet, containing two complex scalars together with two Dirac fermions. The theory we start with is:

<p style="text-align:center">Theory A: $U(1)$ with N charged hypermultiplets</p>

The vortices in this theory fall into the class described in Section 3.5. Each vortex has $2N$ zero modes but, as we discussed, not all of these zero modes are normalizable. The overall center of mass is, of course, normalizable (the vortex has mass $M = 2\pi v^2$) but the remaining $2(N - 1)$ modes of a single vortex are logarithmically divergent.

We now wish to rewrite this theory, treating the vortices as fundamental objects. What properties must the theory have in order to mimic the behavior of the vortex? It will prove useful to think of each vortex as containing N individual "fractional vortices". We postulate that these fractional vortices suffer a logarithmic confining potential, so that any number $n < N$ have a logarithmically divergent mass, but N together form a state with finite mass. Such a system would exhibit the properties of the vortex zero modes described above: the $2N$ zero modes correspond to the positions of the N fractional vortices. They can move happily as a whole, but one pays a logarithmically divergent cost to move these objects individually. (Note: a logarithmically divergent cost isn't really that much!)

In fact, it's very easy to cook up a theory with these properties. In $d = 2 + 1$, an electron experiences logarithmic confinement, since its electric field goes as $E \sim 1/r$ so its energy $\int d^2x \, E^2$ suffers a logarithmic infra-red divergence. These electrons will be our "fractional vortices". We will introduce N different types of electrons and, in order to assure that only bound states of all N are gauge singlets, we introduce $N - 1$ gauge fields with couplings dictated by the quiver diagram shown in the figure. Recall that quiver diagrams are read in the following way: the nodes of the quiver are gauge groups, each giving a $U(1)$ factor in this case. Meanwhile, the links denote hypermultiplets with charge $(+1, -1)$ under the gauge groups to which it is attached. Although there are N

nodes in the quiver, the overall $U(1)$ decouples, leaving us with the theory

Theory B: $U(1)^{N-1}$ with N hypermultiplets

This is the Seiberg-Intriligator mirror theory, capturing the same physics as Theory A. The duality also works the other way, with the electrons of Theory A mapping to the vortices of Theory B. It can be shown that the low-energy dynamics of these two theories exactly agree. This statement can be made precise at the two-derivative level. The Higgs branch of Theory A coincides with the Coulomb branch of Theory B: both are $T^\star(\mathbb{CP}^{N-1})$. Similarly, the Coulomb branch of Theory A coincides with the Higgs branch of Theory A: both are the A_{N-1} ALE space.

There are now many mirror pairs of theories known in three dimensions. In particular, it's possible to tinker with the mirror theories so that they actually coincide at all length scales, rather than simply at low-energies [51]. Mirror pairs for non-abelian gauge theories are known, but are somewhat more complicated to due to presence of instanton corrections (which, recall, are monopoles in three dimensions) [52, 53, 54, 55, 56]. Finally, one can find mirror pairs with less supersymmetry [57, 58], including mirrors for interesting Chern-Simons theories [59, 60, 61]. These Chern-Simons mirrors reduce to Hori-Vafa duality under compactification to two dimensions [62].
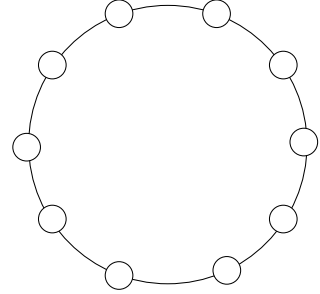
Figure 10:

3.8.3 Vortex Strings

In $d = 3+1$ dimensions, vortices are string like objects. There is a very interesting story to be told about how we quantize vortex worldsheet theory, which is a sigma-model on $\mathcal{V}_{k,N}$. But this will have to wait for the next lecture.

Here let me mention an application of vortices in the context of cosmic strings which shows that reconnection of vortices in gauge theories is inevitable at low-energies. Reconnection of strings means that they swap partners as they intersect as shown in the figure. In general, it's a difficult problem to determine whether reconnection occurs and requires numerical study. However, at low-energies we may reliably employ the techniques of the moduli space approximation that we learnt above [63, 64, 65].

The first step is to reduce the dynamics of cosmic strings to that of particles by considering one of two spatial slices shown in the figure. The vertical slice cuts the strings to reveal a vortex-anti-vortex pair. After reconnection, this slice no longer intersects the strings, implying the annihilation of this pair. Alternatively, one can

slice horizontally to reveal two vortices. Here the smoking gun for reconnection is the right-angle scattering of the vortices at (or near) the interaction point. Such 90º degree scattering is a requirement since, as is clear from the figure, the two ends of each string are travelling in opposite directions after the collision. By varying the slicing along the string, one can reconstruct the entire dynamics of the two strings in this manner and show the inevitability of reconnection at low-energies.
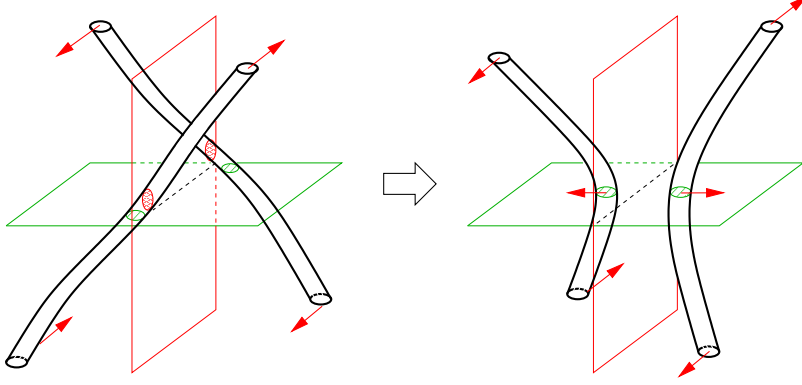

Figure 11: The reconnection cosmic strings. Slicing vertically, one sees a vortex-anti-vortex pair annihilate. Slicing horizontally, one sees two vortices scattering at right angles.

Hence, reconnection of cosmic strings requires both the annihilation of vortex-anti-vortex pairs and the right-angle scattering of two vortices. The former is expected (at least for suitably slow collisions). And we saw in Section 3.3.2 that the latter occurs for abelian vortices in the moduli space approximation. We conclude that abelian cosmic strings do reconnect at low energies. Numerical simulations reveal that these results are robust, holding for very high energy collisions [66].

For cosmic strings in non-abelian theories this result continues to hold, with strings reconnecting except for very finely tuned initial conditions [26]. However, in this case there exist mechanisms to push the strings to these finely tuned conditions, resulting in a probability for reconnection less than 1.

Recently, there has been renewed interest in the reconnection of cosmic strings, with the realization that cosmic strings may be fundamental strings, stretched across the sky [67]. These objects differ from abelian cosmic strings as they have a reduced probability of reconnection, proportional to the string coupling g_s^2 [68, 69]. If cosmic strings are ever discovered, it may be possible to determine their probability of reconnection, giving a

vital clue to their microscopic origin. The recent developments of this story have been nicely summarized in the review [3].

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